J. DIFFERENTIAL GEOMETRY
16 (1981) 185–190

CONVEXITY IN GRAPHS

FRANK HARARY & JUHANI NIEMINEN

The convex hull of a set S of points of a graph G is the smallest set T containing S such that all the points in a geodesic joining two points of T lie in T. The convex hull T can also be formed by taking all geodesics joining two points of S, and iterating that operation. The number of times this is done to S to get T is $\operatorname{gin}(S)$, the geodetic iteration number of S. Then $\operatorname{gin}(G)$ is defined as the maximum of $\operatorname{gin}(S)$ over all sets S of points of G. The smallest number of points in a graph G such that $\operatorname{gin}(G) = n$ is determined and the extremal graphs are constructed.

Let G be a graph with point set $V = V(G)$ and let $S \subset V$. An *S-geodesic* is a shortest path in G joining two points of S. We denote by $(S) = {}^1S$ the set of all points on some S-geodesic. Iterating, let ${}^2S = ({}^1S) = ((S))$ and ${}^{i+1}S = ({}^iS)$. The *geodetic iteration number* of S, written $\operatorname{gin}(S)$, is the minimum n such that ${}^{n+1}S = {}^nS$. Then the *convex hull* of S, denoted by $[S]$, is the point set ${}^nS$. Thus the convex hull of S is the smallest $T \supset S$ such that the points of every T-geodesic are in T.

Trivially $[V] = V$, $[v] = v$ for all $v \in V$, and for each line uv of G, $[u, v] = \{u, v\}$. For other graphical terminology and notation, we follow the book [1]; in particular $p(G)$ is the number of points in G. However, we use E for the set of lines of G. We define the *geodetic iteration number of a graph G* by $\operatorname{gin}(G) = \max\{\operatorname{gin}(S): S \subset V\}$. Our object is to determine the minimum number p of points in a graph G such that $\operatorname{gin}(G)$ has a given value n. Also, the structure of such extremal graphs is specified.

A graph G is *smaller* than graph H if it has fewer points.

Theorem 1. *Let H_n be any smallest graph with geodetic iteration number n. Then the number of points of H_n is given by $p(H_0) = 1$, $p(H_1) = 3$, and when $n \geqslant 2$, $p(H_n) = n + 3$.*

Proof. The case $n = 0$ is trivial and the unique H_0 is the trivial graph K_1. By inspection one sees at once that the extremal graphs H_1 and H_2 are the graphs of Fig. 1 and are unique. We take $S = \{u, v\}$ in both H_1 and H_2 and find that in H_1, $(S) = V$ so that $p(H_1) = 3$, and in H_2, $|(S)| = 4$ and ${}^2S = V$ so we have $p(H_2) = 5 = n + 3$.

Received November 15, 1978.

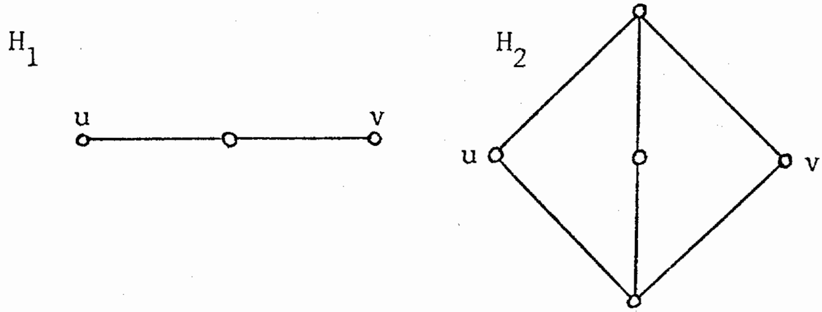


FIG. 1

Now we consider $n \geqslant 3$. By definition there is a nested sequence of point-sets $S = {}^0S \subset {}^1S \subset {}^2S \subset \cdots \subset {}^nS = {}^{n+1}S \subset V$ such that ${}^{i+1}S$ contains ${}^iS$ properly when $0 \leqslant i \leqslant n-1$. Thus a graph G with $\text{gin}(G) = n$ has the minimum number of points if ${}^{i+1}S - {}^iS$ contains only one point for $i = 1, \cdots, n-1$, and if $S = {}^0S$ and ${}^1S$ are of minimum size. If S contains only one point, then as mentioned above $[S] = S$ and thus S must contain at least two points. By the same reasoning ${}^1S - S$ contains at least two points. On the other hand, the graph G of Fig. 2 has $\text{gin}(G) = n$, $S = \{u_0, u_{0*}\}$, ${}^1S - S = \{u_1, u_{1*}\}$ and ${}^{i+1}S - {}^iS = \{u_{i+1}\}$ for $i = 1, \cdots, n-1$. Thus there exists a graph satisfying all the minimum constraints found above, whence $p = n + 3$ in a smallest graph with $\text{gin}(G) = n$ when $n \geqslant 2$.

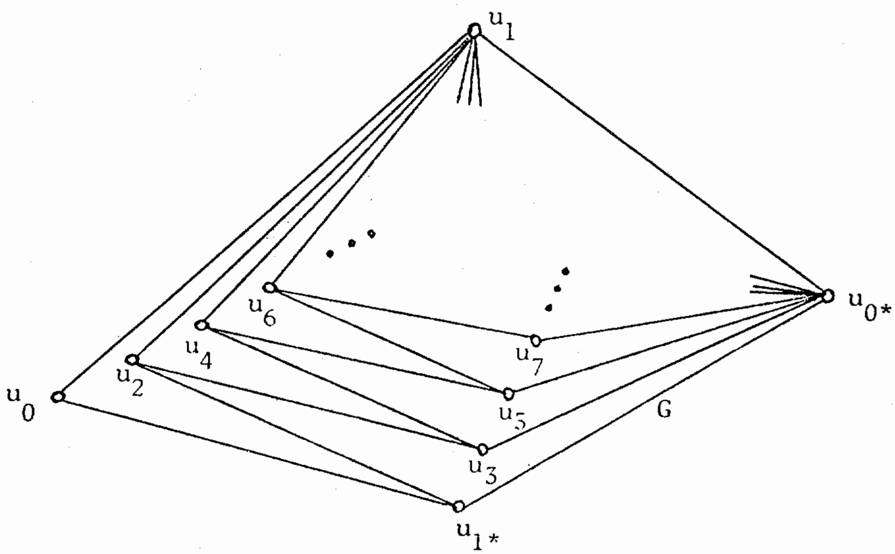


FIG. 2

Theorem 2. *Let G be a graph with* $\text{gin}(G) = n \geqslant 2$, *with a minimum number of points and with point labels* $S = {}^0S = \{u_0, u_{0*}\}$, ${}^1S - S = \{u_1, u_{1*}\}$ *and* ${}^{i+1}S - {}^iS = \{u_{i+1}\}$ *for* $i = 1, \cdots, n-1$. *Then the lines of G satisfy the following requirements*:

(1) $u_0u_1, u_0u_{1*}, u_{0*}u_1, u_{0*}u_{1*}, u_1u_2, u_{1*}u_2 \in E$.

(2) $u_{i+1}u_i, u_{i+1}u_k \in E$ *for each* $i \geqslant 2$ *and for at least one value of k among* $k = 0, 0^*, 1, 1^*, 2, 3, 4, \cdots, i-2$.

(3) *If* $u_{i+1}u_j \in E$ *where* $i \geqslant 3$ *and* $j < i-2$, *then* $u_ju_{j+s} \in E$ *or* $u_{i+1}u_{j+s} \notin E$, $s = 2, 3, \cdots, i-j-1$. *Further, if* $j = 0, 1$, *then* $j \neq 0^*, 1^*$.

Proof. The existence of the lines given in (1) follows from the hypothesis that $\text{gin}(G) \geqslant 2$.

Because $u_{i+1} \in {}^{i+1}S - {}^iS$, there is a geodesic between u_i and another point of ${}^iS$ in G containing u_{i+1}, and as u_{i+1} is the only point in ${}^{i+1}S - {}^iS$, u_{i+1} is joined by a line to two points of ${}^iS$. If $u_{i+1}u_i \notin E$, then $u_tu_{i+1}, u_{i+1}u_r \in E$, where $t, r < i$ and thus $u_t, u_r \in {}^{i-1}S$. When $u_t, u_r \in {}^{i-1}S$, $u_{i+1} \in {}^iS$ which is a contradiction, so every two points of ${}^iS$ adjacent to u_{i+1} in G are joined by a line and thus $u_{i+1} \notin {}^{i+1}S$ which is also a contradiction. Hence (2) is valid.

As $i \geqslant 3$, $u_{i+1} \in {}^1S$ if $j = 0, 0^*$, and $u_{i+1} \in {}^2S$ if $j = 1, 1^*$. Thus the latter statement of (3) holds. By the hypothesis of (3), $u_{i+1}u_j, u_{i+1}u_{j+s} \in E$, so any geodesic between u_j and u_{j+s} is at most of length two. If it has length two, then $u_{i+1} \in {}^{j+s+1}S \subset {}^iS$ which is a contradiction. Thus the length must be one, whence $u_ju_{j+s} \in E$, proving the first part of (3). q.e.d.

In the two next theorems we describe the graphs with $\text{gin}(G) = 0, 1$.

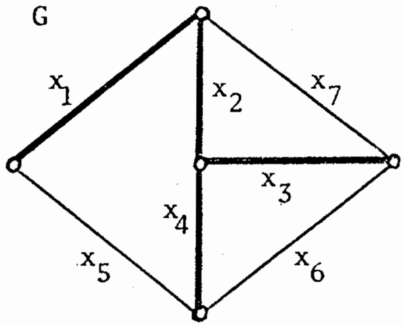
Theorem 3. *A connected graph G has* $\text{gin}(G) = 0$ *if and only if G is a complete graph.*

Proof. Let $\text{gin}(G) = 0$, whence $S = [S]$ for each $S \subset V$. In particular, $S = [S]$ when S contains two points only, and in this case, as G is connected, $S = [S]$ only if the points are adjacent. Hence any two points of G are joined by a line and G is complete. The converse is obvious.

Theorem 4. *Let G be a connected graph. If* $\text{gin}(G) \leqslant 1$, *then there is a cycle basis* $B = \{Z_1, \cdots, Z_k\}$ *of G such that Z_i and Z_j have at most one common line for each pair i and j,* $i \neq j$ *and* $i, j = 1, \cdots, k$.

Proof. If such a cycle basis does not exist, we can choose two cycles Z_i and Z_j of G having minimum number of lines and at least two common lines. By the minimality, if u and v are on the cycles Z_i and Z_j, then all the lines of at least one $\{u, v\}$-geodesic belong to Z_i and Z_j. But then it is easy to choose from the points on Z_i and Z_j a set S such that $S \subset {}^1S \subset {}^2S = {}^3S$, where the points of ${}^2S - {}^1S$ are among the points of the common lines of Z_i and Z_j. Thus $\text{gin}(G) \geqslant 2$, which is a contradiction. q.e.d.

The converse of the theorem does not hold as the graph G of Fig. 3 shows: $\mathrm{gin}(G) = 2$ although there is a cycle basis B satisfying the conditions of Theorem 4.



$$B = \{(x_1, x_2, x_4, x_5),\ (x_2, x_3, x_7),\ (x_3, x_4, x_6)\}.$$

FIG. 3

Finally we look for a criterion for a connected graph G to have $\mathrm{gin}(G) = n$. Some concepts are needed first. Let ${}^nH = \{G\colon \mathrm{gin}(G) = n \text{ and } p = n + 3\}$ when $n \geqslant 2$.

Also let ${}^1H$ consist only of the graph H_1 of Fig. 2 and let ${}^0H = \{K_1\}$. The fact that both ${}^1H$ and ${}^0H$ are singletons was already mentioned above. We shall see that ${}^nH$ is a singleton only for $n = 0, 1, 2, 3, 4$.

The graphs G with $\mathrm{gin}(G) = n$ will be characterized by means of graph homomorphisms and the graphs in the families ${}^nH$ for $n = 0, 1, 2, \cdots$.

Let $G = (V, E)$ be a graph and let $C = \{S_1, \cdots, S_r\}$ be a partition of V. A graph $H = (V_H, E_H)$ is a homomorphic image of G under a homomorphism f, denoted as $f(G) = H$, if there is a one-to-one correspondence between the elements S_j in C and the points u_j in V_H, and if $u_iu_j \in E_H$ whenever there is a line in G joining S_i and S_j, $i \neq j$. We then say that f is *generated* by C. The homomorphism $f\colon G \to H$ is said to be *geodesic compatible* if and only if for each $v - w$ geodesic in G, $v \in S_i$ and $w \in S_j$ and $i \neq j$, ranging over $S_i = S_{i_1}, S_{i_2}, \cdots, S_{i_m} = S_j$, there exists a $u_i - u_j$ geodesic in H ranging over the points $u_i = u_{i_1}, u_{i_2}, \cdots, u_{i_m} = u_j$ and vice versa.

Theorem 5. *For a connected graph G, $\mathrm{gin}(G) = n$ if and only if* (1) *and* (2) *both hold*:

(1) *There are an induced subgraph G' of G and a geodesic compatible homomorphism f such that $f(G') \in {}^nH$.*

(2) *There is no induced subgraph G' and no f as defined in* (1) *such that $f(G') \in {}^mH$, $m > n$.*

Proof. If G satisfies (1), the geodesic compatibility of f implies that $\mathrm{gin}(G) \geqslant n$, and according to (2), $\mathrm{gin}(G) = n$.

We prove the converse by induction on n. When $n = 0$ or 1, the theorem is obviously valid. We assume that the theorem holds for $n \leqslant k$, and let G be a connected graph with $\mathrm{gin}(G) = k + 1$.

As $\mathrm{gin}(G) = k + 1$, there is a set S' such that $S' = {}^0S' \subset {}^1S' \subset {}^2S' \subset \cdots \subset {}^kS' \subset {}^{k+1}S' = [S']$, and as G is connected, $[S']$ obviously induces a connected subgraph of G. According to the properties of a convex hull, ${}^kS'$ also induces a connected subgraph ${}^kG'$ of G. From ${}^{k+1}S' = [S']$ and the induction assumption it follows that by removing points from ${}^0S'$ and ${}^{i+1}S' - {}^iS'$, $i = 0, 1, \cdots, k-1$, we obtain an induced subgraph ${}^kG$ of ${}^kG'$ (and of G) such that

(i) $\mathrm{gin}({}^kG) = k$;

(ii) there is a geodesic compatible homomorphism f' with the property $f'({}^kG) \in {}^kH$;

(iii) there are in ${}^kG$ at least two points joined by a geodesic of G going over the points ${}^{k+1}S' - {}^kS$ in G. As $\mathrm{gin}({}^kG) = k$, there is a sequence $S = {}^0S \subset {}^1S \subset \cdots \subset {}^kS = [S]$, and as the points of the geodesic of (iii) are from ${}^{k+1}S' - {}^kS'$, one of the points of $[S]$ joined by this geodesic is from ${}^kS - {}^{k-1}S$. We denote this point by v, and let $v, w_1, \cdots, w_s, v'$ be a shortest geodesic beginning with v and defined in (iii); thus v and v' are points of ${}^kG$, and $w_1, \cdots, w_s \in {}^{k+1}S' - {}^kS'$. On the other hand, let f' be generated by $C' = \{S_0, S_{0*}, S_1, S_{1*}, S_2, S_3, \cdots, S_k\}$, where ${}^0S = S_0 \cup S_{0*}$, ${}^1S - {}^0S = S_1 \cup S_{1*}$, and $S_j = {}^jS - {}^{j-1}S$, $j = 2, \cdots, k$. A new homomorphism derived from f' is generated by the family $C = C' \cup \{S_{k+1}\}$, where $S_{k+1} = \{w_1, \cdots, w_s\}$. Clearly C is a partition of the points of an induced subgraph ${}^{k+1}G$ of G. We need only show that f is a geodesic compatible homomorphism of ${}^{k+1}G$ onto a graph in ${}^{k+1}H$. According to the properties of f', it is sufficient to concentrate on the set S_{k+1} and its image u_{k+1} in $f({}^{k+1}G)$.

As $v, w_1, \cdots, w_s, v'$ is a shortest geodesic beginning with v and defined in (iii), then only w_s can be adjacent to two or more points of ${}^kG$; in the other case there would be a shorter geodesic beginning with v, which is a contradiction. Let $v' \in S_j$. If there is a line u_ku_j in $f'({}^kG)$, then by removing suitable points from ${}^kS - {}^{k-1}S$, we obtain a new graph ${}^kG$ in which there are no lines joining two points, one from ${}^kS - {}^{k-1}S$ and one from ${}^jS - {}^{j-1}S$. This new ${}^kG$ is connected and satisfies (i) and (iii) as ${}^kS - {}^{k-1}S$ consists of the points of the least iteration in ${}^kG$. As f' is a homomorphism defined in (ii), then *a fortiori* f' is a geodesic compatible homomorphism mapping the new ${}^kG$ onto a graph in ${}^kH$. If w_s is joined by a line to points from other sets S_i than S_j, and u_ku_i is a line in $f'({}^kG)$, then by reducing ${}^kG$ as above, we obtain a new connected graph ${}^kG$ satisfying (i), (ii) and (iii) but in which $f'({}^kG)$ does not contain the line u_ku_i. But then the mapping f of ${}^{k+1}G$ is geodesic compatible, and $f(^{k+1}G)$ contains just those lines which are allowed to belong to a minimum graph with $\mathrm{gin}(G) = k + 1$ in Theorem 2.

If (2) is not valid, then by the first part of the proof, $\mathrm{gin}(G) > k + 1$, which is a contradiction.

Reference

[1] F. Harary, *Graph theory*, Addison-Wesley, Reading, MA, 1969.

UNIVERSITY OF MICHIGAN
UNIVERSITY OF OULU, FINLAND